Int. J. Heat Mass Transfer. Vol. 25, No. 12, pp. 1905–1907, 1982
Printed in Great Britain

0017-9310/82/121905-03 \$03.00/0

SHORTER COMMUNICATIONS

EXAMINATION OF THE TWO-FLUX MODEL FOR RADIATIVE TRANSFER IN PARTICULAR SYSTEMS

M. Q. BREWSTER and C. L. TIEN
Department of Mechanical Engineering, University of California, Berkeley, CA 94720, U.S.A.

(*Received* 2 *November* 1981 *and in final form* 10 *March* 1982)

INTRODUCTION

AN ACCURATE model of the radiative transfer process, including scattering and absorption, is essential to the calculation of radiative heat transfer in many engineering systems. To bridge the gap between the mathematical complexity of exact radiative transfer theory [1] and the need for concise engineering formulas, the two-flux model was developed [2, 3] and has received widespread attention. However, one important consideration has been largely overlooked, namely, the problem of relating the two-flux parameters to fundamental optical properties of the constituent particles, without using any adjustable or empirically-derived constants.

After its initial development [2, 3] and subsequent extension to absorbing–emitting media [4], the two-flux model has further applied to scattering media through the simple, but often unrealistic, assumption of isotropic scattering [5–7]. To account for anisotropic scattering, a variation of the two-flux model, which utilizes two-point Gaussian quadrature [1] was introduced [8, 9]. Anisotropic scattering was also considered in several earlier experimental studies [10–12].

Daniel *et al.* [13] have compared the predictions of the two-flux model with results of exact theory for radiation in shallow ponds and concluded that, at least in that situation, the two-flux model was unsuitable. The main disadvantage of the two-flux model, it was noted, is its lack of ability to represent the step-change of refractive index at the air–water interface, which gives rise to a rather pronounced anisotropic distribution of intensity due to total internal reflection of rays past the critical angle. Since not all engineering systems exhibit boundaries of this kind, the utility of the two-flux model has by no means been discredited in general. It is also important to note that the conclusions in ref. [13] were based on local volumetric absorption values rather than global transmittance or reflectance values, which tend to have more significance for heat transfer applications.

More recently, Tong and Tien [14] have attempted to use only basic constituent properties and the electromagnetic scattering theory to predict the two-flux parameters, including the two-flux scattering and absorption coefficients as well as the back-scatter fraction, for thermal radiation in fibrous insulation. However, the expression for the back-scatter fraction was taken from a frequently-encountered alternative development of the two-flux model [15], which does not relate the back-scatter fraction to the single scattering phase function in a manner consistent with the semi-isotropic assumption of the original two-flux model. In addition to the two-flux model, the linear anisotropic model of Dayan and Tien [16] offers another approximate solution of the complete transfer problem. Tong and Tien [17] have compared calculations of the linear anisotropic model with those of the two-flux model and concluded that although intermediate parameters may sometimes vary appreciably between the two, the total predicted heat flux is usually very nearly the same for both cases.

In this study, an attempt is made to assess the predictive capability of the two-flux model without relying on any empirical or adjustable constants. Predictions of the two-flux model are compared with those of exact radiative transfer theory to determine the influence of optical depth and particle size parameter on the accuracy of the two-flux model.

ANALYSIS

The traditional two-flux model, based on the assumption of semi-isotropic intensity distribution, is obtained by integrating the complete transfer equation for azimuthally symmetric radiation in a 1-dim., plane-parallel slab [1],

$$\mu \frac{\partial I}{\partial z}(z,\mu) = -(\sigma + a)I + \frac{\sigma}{2}\int_{-1}^{1} I(x,\mu')p(\mu,\mu')\mathrm{d}\mu' \quad (1)$$

over all directions (4π solid angle), giving

$$\frac{\mathrm{d}I^+}{\mathrm{d}z} = -(\bar{\sigma} + \bar{a})I^+ + \bar{\sigma}I^-, \quad (2)$$

$$-\frac{\mathrm{d}I^-}{\mathrm{d}z} = -(\bar{\sigma} + \bar{a})I^- + \bar{\sigma}I^+. \quad (3)$$

For the nomenclature, μ is the directional cosine with respect to coordinate z perpendicular to the slab, I the radiation intensity, σ the scattering coefficient, a the absorption coefficient, and p the scattering phase function. The superscripts + and − refer to forward ($\mu > 0$) and backward ($\mu < 0$) directions, respectively. The bar quantities denote hemispherically integrated values, in accordance with the two-flux model.

The two-flux scattering and absorption coefficients can be expressed as

$$\bar{\sigma} = 2B\sigma = 3BQ_s f_v/d; \; \bar{a} = 2a = 3Q_a f_v/d \quad (4)$$

where Q_s and Q_a are the single particle scattering and absorption efficiencies, d is the particle diameter, and f_v represents the particle volume fraction. The third parameter of the two-flux model, the back-scatter fraction B is formally defined as

$$B = \frac{1}{2}\int_0^1 \int_{-1}^0 p(\mu,\mu')\mathrm{d}\mu'\,\mathrm{d}\mu \quad (5)$$

where the phase function for the 1-dim. plane-parallel slab is related to single-particle scattering phase function $p(\theta)$ [1] as:

$$p(\mu,\mu') = \frac{1}{\pi}\int_0^{\pi} p[\theta(\mu,\psi;\mu',\psi = 0)]\,\mathrm{d}\psi \quad (6)$$

with θ being the angle between the incidence and the scattering directions for the spherical particle. Equation (5) was evaluated by numerical integration [18] of the Mie scattering solution as a function of the size parameter, $x = \pi d/\lambda$, with λ being the wavelength. Because of the large values

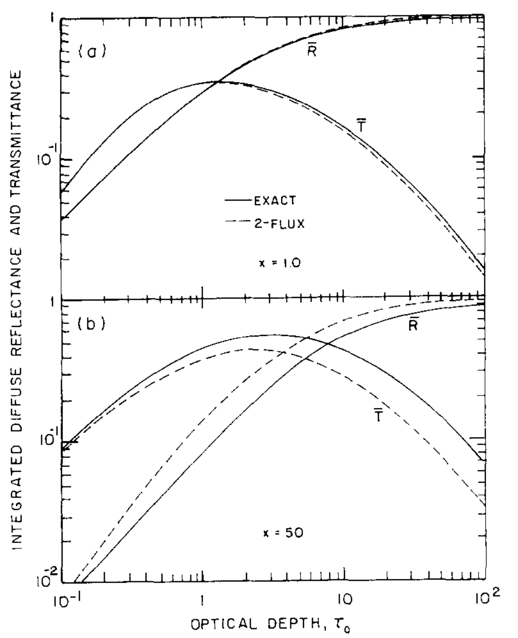

FIG. 1. Two-flux predictions vs optical depth for $n = 1.21$ and $\omega_0 = 1.0$: (a) $x = 1.0$; (b) $x = 50$.

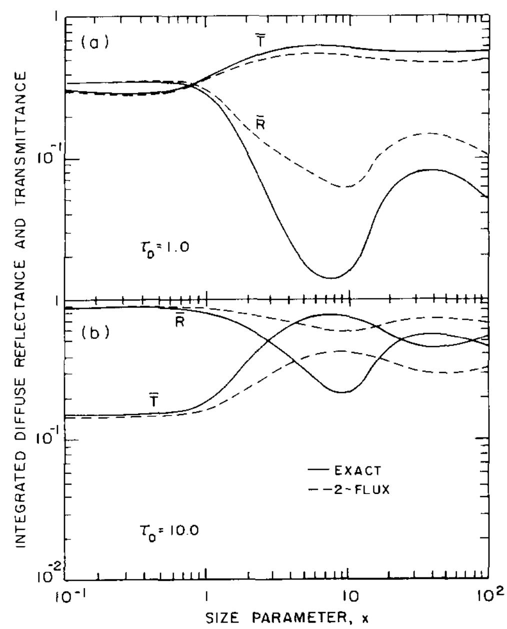

FIG. 2. Two-flux predictions vs size parameter for $n = 1.21$ and $\omega_0 = 1.0$: (a) $\tau_0 = 1.0$; (b) $\tau_0 = 10.0$.

of x covered, it was not particularly convenient in this case to utilize a Legendre polynomial expansion for $p(\theta)$ with exact Gaussian quadrature as is customarily done with systems of 'smaller' particles ($x \sim 1$) which have quasi-isotropic phase functions. To do so and still cover a broad range of values for x would result in a prohibitively large number of terms in the expansions for $p(\theta)$. Naturally the numerical integration was verified to make certain that numerical errors did not inadvertantly lead to violation of conservation of energy.

The form of the two-flux equations appearing in equations (2) and (3) is suitable for a slab with diffuse incidence. Since the calculations presented here were performed in conjunction with an experimental study [18, 19] which employed collimated incidence and reflecting boundaries, the two-flux equations were recast in terms of the scattered component of intensity with the collimated unscattered component appearing as a source term in the equations [1]. The solution of the resulting simultaneous equations is straightforward. For exact results the method of discrete ordinates was applied to solve the transfer equation. A complete discussion of this method can be found elsewhere [18, 20].

RESULTS AND DISCUSSION

It has been suggested that the two-flux model is invalid when the single scattering phase function is strongly anisotropic [13]. It has also been suggested [21] that the two-flux equations need to be multiplied by a constant $2/\sqrt{3}$ in order to agree with the optically thick diffusion approximation, i.e. $\tau_0 \gg 1$ where $\tau_0 = (\sigma + a)L$ and L is the slab thickness. To characterize the influence of the optical depth τ_0 relative to that of the phase function on the reliability of the two-flux model, comparative calculations were made while systematically varying the values of τ_0 and x. Figures 1 and 2 demonstrate these results.

In Fig. 1(a), τ_0 is varied from 0.1 to 100 while x is fixed at a value of one. The phase function is rather moderate ($B = 0.43$) and the two-flux results are in error by at most 10% over the entire range of τ_0. In Fig. 1(b), however, for the same region of τ_0 but for $x = 50$, the slab reflectance is over-predicted by as much as 40% (even at smaller τ_0) and transmittance is under-predicted by approx. 30–50% for $\tau_0 \gtrsim 2.0$. It should be noted that the main difference between these two cases is that for $x = 50$, the scattering becomes very strong in the forward direction ($B = 0.14$).

On the other hand, Fig. 2 shows transmittance and reflectance plotted as a function of x, with τ_0 as a parameter. In Fig. 2(a), $\tau_0 = 1.0$ and in Fig. 2(b), $\tau_0 = 10.0$. In both cases, now, it is evident that as x increases beyond the value of one, the accuracy of the two-flux model diminishes substantially. The significance of large values of x is that B decreases markedly for x greater than one.

The dips in the values of $\bar{R}$ near $x = 10$ are a result of the fact that as $x \to \infty$ the strong forward spike in the phase function which eventually accounts for the Fresnel diffraction pattern must be removed from consideration as scattered energy in order for the method of discrete ordinates to predict accurate results for large values of x. For $x > 10$, the forward scattering spike was removed from the phase function, corresponding to the limit $Q_s \to 1$ as $x \to \infty$. For $x < 10$, the usual method was followed with both the numerical integration of $p(\theta)$ and the Legendre polynomial techniques producing exactly the same results. This means, of course, that there is uncertainty in the accuracy of even the method of discrete ordinates in the transition region between the Mie range ($x \sim 1$) and the geometric limit ($x \to \infty$) and this problem is now being considered.

SUMMARY

Predictions of radiative transfer using the two-flux model have been compared with those of exact theory. No adjustable coefficients have been employed and thus, the predictive capability of the two-flux model using only fundamental properties of the system has been examined. Although similar studies have been conducted in the past, it has never been made clear whether acute anisotropic single scattering or large optical thickness or both conditions together are mainly responsible for disagreement with exact radiative transfer theory. It is shown here that acute single scattering anisotropy is the prime cause in inaccuracy in the two-flux model. Although the combination of large optical thickness with strong forward scattering is the worst case, for quasi-isotropic phase functions, the predictions of the two-flux model agree quite well with exact theory even at large optical depths ($\tau_0 \sim 100$). In order to obtain accurate results from fundamental

properties (without adjustable constants) when scattering is acutely anisotropic, it is necessary to resort to either exact numerical methods or one of the more complicated approximate models (three-flux, six-flux, etc.).

REFERENCES

1. S. Chandrasekhar, *Radiative Transfer*. Dover, New York (1960).
2. A. Schuster, Radiation through a foggy atmosphere, *Astrophys. J.* **217,** 1–22 (1905).
3. K. Schwarzchild, Equilibrium of the Sun's atmosphere, *Ges. Wiss. Gottingen, Nachr., Math.-Phys. Klasse* **1,** 41–53 (1906).
4. H. C. Hamaker, Radiation and heat conduction in light scattering material, *Philips Res. Rep.* **2,** 55–67, 103–111, 112–125, 420 (1947).
5. K. Klier, Absorption and scattering in plane parallel turbid media, *J. Opt. Soc. Am.* **62,** 882–885 (1948).
6. P. Kubelka, New contribution to the optics of intensity light-scattering materials, Part I, *J. Opt. Soc. Am.* **38,** 448–457 (1948).
7. B. F. Armaly and T. T. Lam, Influence of refractive index on reflectance from a semi-infinite absorbing-scattering medium with collimated incident radiation, *Int. J. Heat Mass Transfer* **18,** 893–899 (1975).
8. C. Sagan and J. Pollack, Anisotropic nonconservative scattering and the clouds of Venus, *J. Geophys. Res.* **72,** 469–474 (1967).
9. G. Domoto and W. Wang, Radiative transfer in homogeneous non-gray with nonisotropic particle scattering, *J. Heat Transfer* **96,** 385–390 (1974).
10. B. K. Larkin and S. W. Churchill, Heat transfer by radiation through porous insulations, *A.I.Ch.E. Jl* **5,** 467–473 (1959).
11. J. C. Chen and S. W. Churchill, Radiant heat transfer in packed beds, *A.I.Ch.E. Jl* **9,** 35–41 (1963).
12. C. F. Sanders and J. M. Lenoir, Radiative transfer through a cloud of absorbing-scattering particles, *A.I.Ch.E. Jl* **18,** 155–160 (1972).
13. K. J. Daniel, N. M. Laurendeau and F. P. Incropera, Prediction of radiation absorption and scattering in turbid water bodies, *J. Heat Transfer* **101,** 63–67 (1979).
14. T. W. Tong and C. L. Tien, Analytical models for thermal radiation in fibrous insulations, *J. Thermal Insulation* **4,** 27–44 (1980).
15. H. C. Hottel and A. F. Sarofim, *Radiative Transfer*. McGraw-Hill, New York (1967).
16. A. Dayan and C. L. Tien, Radiative transfer with anisotropic scattering in an isothermal slab, *Jl Quantve Spectros. & Radiat. Transf.* **16,** 113–125 (1976).
17. T. W. Tong and C. L. Tien, Resistance network representation of radiative heat transfer with particulate scattering, *Jl Quantve Spectros. & Radiat. Transf.* **24,** 491–503 (1980).
18. Q. Brewster, Radiative transfer in packed and fluidized beds, Ph.D. Dissertation in Mechanical Engineering, University of California, Berkeley (1981).
19. M. Q. Brewster and C. L. Tien, Radiative transfer in packed/fluidized beds: dependent vs. independent scattering, *J. Heat Transfer*, to be published (1982).
20. H. C. Hottel, A. F. Sarofim, L. B. Evans and I. A. Vasalos, Radiative transfer in anisotropically scattering media: allowance for fresnel reflection at the boundaries, *J. Heat Transfer* **90,** 56–62 (1968).
21. R. Viskanta, A. Ungan and M. P. Menguc, Predictions of radiative properties of pulverized coal and fly-ash polydispersions, ASME Paper 81-HT-24 (1981).

Int. J. Heat Mass Transfer. Vol. 25, No. 12, pp. 1907–1909, 1982
Printed in Great Britain
0017-9310/82/121907-03 $03.00/0

A STEADY STATE LINEAR ABLATION PROBLEM

G. J. Weir

Applied Mathematics Division, Department of Scientific and Industrial Research, Wellington, New Zealand

(*Received* 27 *January* 1982)

NOMENCLATURE

h,	heat transfer coefficient;
r,	rod radius;
r',	radial coordinate;
r_s,	maximum ablation radius, equation (12);
s,	rod resistivity;
z,	axial coordinate;
C,	rod heat capacity;
I,	current;
I_s,	maximum rod current, equation (11);
J,	energy flux;
K,	rod thermal conductivity;
T,	temperature;
$T(0)$,	facial temperature;
T_0,	external temperature;
T_s,	sublimation temperature;
T_∞,	asymptotic rod temperature;
U,	rod speed;
U_s,	maximum rod velocity, equation (10);
V,	potential drop;
V_∞,	potential for constant rod temperature.

Greek symbols

λ,	latent heat of vaporisation;
μ,	non-ablating inverse distance;
ν,	ablating inverse distance;
ρ,	rod density;
ϕ,	angular coordinate.

INTRODUCTION

Consider a semi-infinite cylindrical rod of radius r lying along the positive z axis of a cylindrical coordinate system (r', ϕ, z). If a uniform distribution of current of I A enters the rod